Asymptotic Behavior of The Maximum Likelihood Estimator For Ergodic and Nonergodic Sqaure-Root Diffusions

Mohamed Ben Alaya, Ahmed Kebaier

Asymptotic Behavior of The Maximum Likelihood Estimator For Ergodic and Nonergodic Sqaure-Root Diffusions

By Mohamed Ben Alaya and Ahmed Kebaier*

LAGA, CNRS (UMR 7539), Institut Galilée, Université Paris 13,
99, av. J.B. Clément 93430 Villetaneuse, France
mba@math.univ-paris13.fr kebaier@math.univ-paris13.fr

November 10, 2011

Abstract

This paper deals with the problem of global parameter estimation in the Cox-Ingersoll-Ross (CIR) model $(X_t)_{t\geq 0}$. This model is frequently used in finance for example to model the evolution of short-term interest rates or as a dynamic of the volatility in the Heston model. In continuity with a recent work by Ben Alaya and Kebaier [1], we establish new asymptotic results on the maximum likelihood estimator (MLE) associated to the global estimation of the drift parameters of $(X_t)_{t\geq 0}$. To do so, we need to study first the asymptotic behavior of the quadruplet $(\log X_t, X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$. This allows us to obtain various and original limit theorems on our MLE, with different rates and different types of limit distributions. Our results are obtained for both cases : ergodic and nonergodic diffusion.

AMS 2000 Mathematics Subject Classification. 44A10, 60F05, 62F12.

Key Words and Phrases. Cox-Ingersoll-Ross processes, nonergodic diffusion, Laplace transform, limit theorems, parameter inference.

1 Introduction

The Cox-Ingersoll-Ross (CIR) process is widely used in mathematical finance to model the evolution of short-term interest rates. It is also used in the valuation of interest rate derivatives.

*This research benefited from the support of the chair "Risques Financiers", Fondation du Risque.

It was introduced by Cox, Ingersoll and Ross [2] as solution to the stochastic differential equation (SDE)

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma|X_t|}dW_t, \tag{1}$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion. Under the above assumption on the parameters that we will suppose valid through all the paper, this SDE has a unique non-negative strong solution $(X_t)_{t\geq 0}$ (see Ikeda and Watanabe [8], p. 221). In the particular case $b = 0$ and $\sigma = 2$, we recover the square of a a-dimensional Bessel process starting at x. For extensive studies on Bessel processes we refer to Revuz and Yor [16] and Pitman and Yor [14] and [15]. The behavior of the CIR process X mainly depends on the sign of b. Indeed, in the case $b > 0$ there exists a unique stationary distribution, say π, of X and the stationary CIR processes enjoy the ergodic property that is: for all $h \in L^1(\pi)$, $\frac{1}{t}\int_0^t h(X_s)ds$ converges almost surely to $\int_{\mathbb{R}} h(x)\pi(dx)$. In the case $a \geq \sigma$, the CIR process X stays strictly positive; for $0 < a < \sigma$, it hits 0 with probability $p \in]0, 1[$ if $b < 0$ and almost surely if $b \geq 0$, the state 0 is instantaneously reflecting (see e.g. Göing-Jaeschke and Yor [6] for more details).

During the past decades, inference for diffusion models has become one of the core areas in statistical sciences. The basic results concerning the problem of estimating the drift parameters when a diffusion process was observed continuously are well-summarized in the books by Lipster and Shiryayev [13] and Kutoyants [11]. This approach is rather theoretical, since the real data are discrete time observations. Nevertheless, if the error due to discretization is negligible then the statistical results obtained for the continuous time model are valid for discrete time observations too. Most of these results concern the case of ergodic diffusions with coefficients satisfying the Lipschitz and linearity growth conditions. In the literature, only few results can be found for nonergodic diffusions or diffusions with nonregular coefficients such as the CIR. To our knowledge, one of the first papers having studied the problem of global parameter estimation in the CIR model is that of Fournié and Talay [4]. They have obtained a nice explicit formula of the maximum likelihood estimator (MLE) of the drift parameters $\theta := (a, b)$ and have established its asymptotic normality only in the case $b > 0$ and $a > \sigma$ by using the classical martingale central limit theorem. Otherwise, for cases $b \leq 0$ or $a \leq \sigma$ this argument is no more valid. Note that, in practice, the parameter σ is usually assumed to be known and one can estimate it separately using the quadratic variation of the process X.

In a recent work Ben Alaya and Kebaier [1] use a new approach, based on Laplace transform technics, to study the asymptotic behavior of the MLE associated to one of the drift parameters in the CIR model, given that the other one is known, for a range of values (a, b, σ) covering ergodic and nonergodic situations. More precisely, they prove that

- **First case:** when a is supposed to be known the MLE of $\theta = b$ is given by

$$\hat{b}_T = \frac{aT + x - X_T}{\int_0^T X_s ds}.$$

  If $b > 0$, the asymptotic theorem on the error $\hat{b}_T - b$ is obtained with a rate equal to $\sqrt{T}$ and the limit distribution is Gaussian. However, for $b < 0$ and $b = 0$, when the diffusion $(X_t)_{t\geq 0}$ is not ergodic, the rate of convergence are respectively equal to $e^{-bT/2}$ and T and the obtained limit distributions are not guassians (for more details see Theorem 1 of [1])

- **Second case:** when b is supposed to be known, the MLE of $\theta = a$, given by

$$\hat{a}_T = \frac{\log X_T - \log x + bT + \sigma \int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}},$$

is well defined only for $a \geq \sigma$. When $a > \sigma$, the asymptotic theorem on the error $\hat{a}_T - a$ is obtained with a rate equal to $\sqrt{T}$ if $b > 0$ and with a rate equal to $\sqrt{\log T}$ if $b = 0$ and the limit distribution is Gaussian in both cases. When $a = \sigma$, the asymptotic theorem on the error $\hat{a}_T - a$ is obtained with a rate equal to T if $b > 0$ and with a rate equal to $\log T$ if $b = 0$ and the corresponding limit distributions are not guassians. However, for the case $b < 0$ the MLE estimator $\hat{a}_T$ is not consistent (for more details see Theorem 2 of [1]). Note that for the case $b \leq 0$, when the diffusion $(X_t)_{t\geq 0}$ is not ergodic, and even in the case $b > 0$ and $a = \sigma$, when the diffision is ergodic, the quantity $\frac{1}{t}\int_0^t \frac{1}{X_s} ds$ goes to infinity as t tends to infinity and consequently the classical technics based on central limt theorem for martingales fail.

In order to pove the first case results, Ben Alaya and Kebaier [1] use the explicit Laplace transform of the couple $(X_T, \int_0^T X_s ds)$ which is well known (see e.g. Lamberton and Lapeyre [12], p. 127), in view to study its asymptotic behavior (see Theorem 1). However, for the second case, they only use the explicit Laplace transform of $\int_0^T \frac{ds}{X_s}$ based on a recent work of Craddock and Lennox [3] to deduce its asymptotic behavior too (see Theorem 3).

The aim of this paper is to study the global MLE estimator of parameter $\theta = (a, b)$ for a range of values (a, b, σ) covering ergodic and nonergodic situations. Indeed, it turns out that the MLE of $\theta = (a, b)$ is defined only when $a \geq \sigma$ and the associated estimation error is given by

$$\hat{\theta}_T - \theta = \begin{cases} \hat{a}_T - a & = & \dfrac{\left(\log X_T - \log x + (\sigma - a)\int_0^T \frac{ds}{X_s}\right)\int_0^T X_s ds - T\left(X_T - x - aT\right)}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \\ \hat{b}_T - b & = & \dfrac{T\left(\log X_T - \log x + bT + \sigma\int_0^T \frac{ds}{X_s}\right) - \left(X_T - x + b\int_0^T X_s ds\right)\int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \end{cases}.$$

Hence, the results obtained by [1] are no longer sufficient for our proposed study, since we need a precise description of the asymptotic behavior of the quadruplet $(\log X_T, X_T, \int_0^T X_s ds, \int_0^T \frac{ds}{X_s})$ appearing in the above error estimation. For this purpose, we use once again the recent work of [3]. The obtained results are stated in the second section.

Then in the third section, we take advantage of this study to establish new asymptotic theorems on the error estimation $\hat{\theta}_T - \theta$ with different rates of convergence and different types of limit distributions that vary according to assumptions made on the parameters a, b and σ. More precisely, we prove that when $a > \sigma$, the asymptotic theorem on the error $\hat{\theta}_T - \theta$ is obtained with a rate equal to $\sqrt{T}$ if $b > 0$ and with a rate equal to $\mathrm{diag}(\sqrt{\log T}, T)$ if $b = 0$ and the limit distribution is Gaussian only for the first case. When $a = \sigma$, the asymptotic theorem on the error $\hat{\theta}_T - \theta$ is obtained with a rate equal to $\mathrm{diag}(T, \sqrt{T})$ if $b > 0$ and with a rate equal to $\mathrm{diag}(\log T, T)$ if $b = 0$ and the corresponding limit distributions are not guassians. For $b < 0$ the MLE estimator $\hat{\theta}_T$ is not consistent. Our results cover both cases : ergodic and nonergodic situations and are summarized in Theorems 5, 6 and 7.

In the last section we study the problem of parameter estimation from discrete observations. These observations consist of a discrete sample $(X_{t_k})_{0\le k\le n}$ of the CIR diffusion at deterministic and equidistant instants $(t_k = k\Delta_n)_{0\le k\le n}$. Our aim is to study a new estimator $\hat{\theta}^{\Delta_n}_{t_n} := (\hat{a}^{\Delta_n}_{t_n}, \hat{b}^{\Delta_n}_{t_n})$, for $\theta = (a, b)$ based on discrete observations, under the conditions of high frequency, $\Delta_n \to 0$, and infinite horizon, $n\Delta_n \to \infty$. Several authors have studied this estimation problem by basing the inference on a discretization of the continuous likelihood ratio, see Genon-Catalot [5] and Yoshida [17]. In our approach, we proceed in a slightly different way, we discretize the continuous time MLE instead of considering the maximum argument of the discretized likelihood. We give sufficient conditions on the stepsize Δ_n under which the error $\hat{\theta}^{\Delta_n}_{t_n} - \hat{\theta}_{t_n}$ correctly normalized tends to zero, so that the limit theorems obtained in the continuous time observations for $\hat{\theta}_{t_n}$ can be easily carried out for the discrete one $\theta^{\Delta_n}_{t_n}$. It turns out that, for $a > 2\sigma$, these limit theorems are satisfied if $n^2\Delta_n \to 0$, when $b > 0$, or $\max\left(n^2\Delta_n, \frac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)}\right) \to 0$, when $b = 0$ (see Theorems 8 and 9). The case $0 \le a \le 2\sigma$, seems to be much more harder to treat and requires a specific and more detailed analysis see the last Remark of section 4. For $b > 0$, the condition $n^2\Delta_n \to 0$ is consistent with those of papers in the litterature dealing with the same problem for ergodic diffusions with regular coefficients (see Yoshida [17] and its references). However, for $b = 0$, the condition $\max\left(n^2\Delta_n, \frac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)}\right) \to 0$ seems to be quite original since it concerns a nonergodic case.

2 Asymptotic behavior of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$

Let us recall that $(X_t)_{t\ge0}$ denotes a CIR process solution to (1). It is relevant to consider separately the cases $b = 0$ and $b \neq 0$, since the process $(X_t)_{t\ge0}$ behaves differently.

2.1 Case $b = 0$

In this section, we consider the Cox-Ingersoll-Ross CIR process $(X_t)_{t\ge0}$ with $b = 0$. In this particular case, $(X_t)_{t\ge0}$ satisfies the SDE

$$dX_t = adt + \sqrt{2\sigma X_t}dW_t. \tag{2}$$

Note that for $\sigma = 2$, we recover the square of a a-dimensional Bessel process starting at x and denoted by BESQ^a_x. This process has been attracting considerably the attention of several studies (see Revuz and Yor [16]). Recently, Ben Alaya and Kebaier study the seperate asymptotic behavior of $(X_t, \int_0^t X_s ds)$ and $\int_0^t \frac{ds}{X_s}$ (see Propositions 1 and 2 of [1]) and prove the following result

Theorem 1 *Let $(X_t)_{t\ge0}$ be a CIR process solution to (2), we have*

1. $(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds) \xrightarrow{law} (R_1, I_1)$ *as* t *tends to infinity, where* $(R_t)_{t\geq 0}$ *is the CIR process starting from* 0*, solution to (2) and* $I_t = \int_0^t R_s ds$.

2. $\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = 1$ *if and only if* $a \geq \sigma$.

3. *If* $a > \sigma$ *then* $\frac{1}{\log t}\int_0^t \frac{ds}{X_s} \xrightarrow{\mathbb{P}} \frac{1}{a-\sigma}$ *as* t *tends to infinity.*

4. *If* $a = \sigma$ *then* $\frac{1}{(\log t)^2}\int_0^t \frac{ds}{X_s} \xrightarrow{law} \tau_1$ *as* t *tends to infinity, where* τ_1 *is the hitting time associated with Brownian motion* $\tau_1 := \inf\{t > 0 : W_t = \frac{1}{\sqrt{2\sigma}}\}$.

In the following, we study the asymptotic behavior of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$ which makes sense only and olny if $a \geq \sigma$ and we prove the following result.

Theorem 2 *Let* $(X_t)_{t\geq 0}$ *be a CIR process solution to (2), we have*

1. *If* $a > \sigma$ *then*

$$(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds, \frac{1}{\log t}\int_0^t \frac{ds}{X_s}) \xrightarrow{law} (R_1, I_1, \frac{1}{a-\sigma}) \text{ as } t \text{ tends to infinity.}$$

2. *If* $a = \sigma$ *then*

$$(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds, \frac{1}{(\log t)^2}\int_0^t \frac{ds}{X_s}) \xrightarrow{law} (R_1, I_1, \tau_1) \text{ as } t \text{ tends to infinity.}$$

Here, $(R_t)_{t\geq 0}$ *denotes the CIR process starting from* 0*, solution to (2),* $I_t = \int_0^t R_s ds$*, and* τ_1 *is the hitting time associated with Brownian motion* $\tau_1 := \inf\{t > 0 : W_t = \frac{1}{\sqrt{2\sigma}}\}$*. Moreover, The couple* (R_1, I_1) *and the random time* τ_1 *are independent.*

In order to prove this proposition, we choose to compute the Laplace transform of the quadruplet $(\log X_t, X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$. Here is the obtained result.

Proposition 1 *For* $\rho \geq 0, \lambda \geq 0$, $\mu > 0$ *and* $\eta \in]-k-\frac{\nu}{2}-\frac{1}{2}, +\infty[$*, we have*

$$\begin{aligned}\mathbb{E}_x\left(X_t^\eta e^{-\rho X_t - \lambda\int_0^t X_s ds - \mu\int_0^t \frac{ds}{X_s}}\right) &= \frac{\Gamma(\eta + k + \frac{\nu}{2} + \frac{1}{2})}{\Gamma(\nu+1)} x^\eta \exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma}\coth(\sqrt{\sigma\lambda}t)\right) \\ &\times \left(\frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)}\right)^{\frac{\nu}{2}+\frac{1}{2}-k-\eta} \left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t) + \cosh(\sqrt{\sigma\lambda}t)\right)^{-\frac{\nu}{2}-\frac{1}{2}-k-\eta} \\ &\times {}_1F_1\left(\eta + k + \frac{\nu}{2} + \frac{1}{2}, \nu+1, \frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)\left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t) + \cosh(\sqrt{\sigma\lambda}t)\right)}\right) \quad (3)\end{aligned}$$

where $k = \dfrac{a}{2\sigma}$, $\nu = \dfrac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$ *and* ${}_1F_1$ *is the confluent hypergeometric function defined by* ${}_1F_1(u, v, z) = \sum_{n=0}^{\infty} \frac{u_n}{v_n}\frac{z^n}{n!}$, *with* $u_0 = v_0 = 1$, *and for* $n \geq 1$, $u_n = \prod_{k=0}^{n-1}(u+k)$ *and* $v_n = \prod_{k=0}^{n-1}(v+k)$.

Consequently, the Laplace transform of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$ is obtained by taking $\eta = 0$ in relation (3) and we have the following result.

Corollary 1 *For* $\rho \geq 0, \lambda \geq 0$ *and* $\mu > 0$, *we have*

$$\begin{aligned}
\mathbb{E}_x\left(e^{-\rho X_t - \lambda\int_0^t X_s ds - \mu\int_0^t \frac{ds}{X_s}}\right) &= \frac{\Gamma(k+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\nu+1)}\exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma}\coth(\sqrt{\sigma\lambda}t)\right) \\
&\times \left(\frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)}\right)^{\frac{\nu}{2}+\frac{1}{2}-k}\Big((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t)+\cosh(\sqrt{\sigma\lambda}t)\Big)^{-\frac{\nu}{2}-\frac{1}{2}-k} \\
&\times {}_1F_1\left(k+\frac{\nu}{2}+\frac{1}{2}, \nu+1, \frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)\Big((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t)+\cosh(\sqrt{\sigma\lambda}t)\Big)}\right)
\end{aligned} \quad (4)$$

where $k = \dfrac{a}{2\sigma}$, $\nu = \dfrac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$.

Proof of Proposition 1: We apply Theorem 5.10 in [3] to our process. We have just to be careful with the misprint in formula (5.24) of [3]. More precisely, we have to replace $\sqrt{Axy}$, in the numerator of the first term in the right hand side of this formula, by $\sqrt{Ax/y}$. Hence, for $a > 0$ and $\sigma > 0$, we obtain the so called fundamental solution of the PDE $u_t = \sigma x u_{xx} + a u_x - (\frac{\mu}{x} + \lambda x)u, \quad \lambda > 0, \mu > 0$:

$$p(t,x,y) = \frac{\sqrt{\sigma\lambda}}{\sigma\sinh(\sqrt{\sigma\lambda}t)}\left(\frac{y}{x}\right)^{k-1/2}\exp\left(-\frac{\sqrt{\sigma\lambda}(x+y)}{\sigma\tanh(\sqrt{\sigma\lambda}t)}\right) I_\nu\left(\frac{2\sqrt{\sigma\lambda}\sqrt{xy}}{\sigma\sinh(\sqrt{\sigma\lambda}t)}\right), \quad (5)$$

where I_ν is the modified Bessel function of the first kind. This yields the Laplace transform of the quadruplet $(\log X_t, X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$, since

$$\mathbb{E}_x\left(X_t^\eta e^{-\rho X_t - \lambda\int_0^t X_s ds - \mu\int_0^t \frac{ds}{X_s}}\right) = \int_0^\infty y^\eta e^{-\rho y} p(t,x,y)dy.$$

Evaluation of this integral is routine, see formula 2 of section 6.643 in [7]. Therefore, we get

$$\begin{aligned}
\mathbb{E}_x\left(X_t^{\eta}e^{-\rho X_t-\lambda\int_0^t X_s ds-\mu\int_0^t \frac{ds}{X_s}}\right) &= \frac{\Gamma(\eta+k+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\nu+1)}x^{-k}\exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma}\coth(\sqrt{\sigma\lambda}t)\right) \\
&\times\left(\frac{\sigma\sinh(\sqrt{\sigma\lambda}t)}{\sqrt{\sigma\lambda}\left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t)+\cosh(\sqrt{\sigma\lambda}t)\right)}\right)^{\eta+k} \\
&\times\exp\left(\frac{\sqrt{\sigma\lambda}x}{2\sigma\sinh(\sqrt{\sigma\lambda}t)\left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t)+\cosh(\sqrt{\sigma\lambda}t)\right)}\right) \\
&\times M_{-\eta-k,\frac{\nu}{2}}\left(\frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)\left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t)+\cosh(\sqrt{\sigma\lambda}t)\right)}\right), \qquad (6)
\end{aligned}$$

where $M_{s,r}(z)$ is the Whittaker function of the first kind given by

$$M_{s,r}(z) = z^{r+\frac{1}{2}}e^{-\frac{z}{2}}\ {}_1F_1(r-s+\frac{1}{2}, 2r+1, z). \qquad (7)$$

See [7] for more details about those special functions. We complete the proof by inserting relation (7) in (6). □

Proof of Theorem 2 : The first assertion is straightforward from Theorem 1. For the last assertion and under notations ofthe above corollary, it is easy to check that

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}_x\left(e^{-\frac{\rho}{t}X_t-\frac{\lambda}{t^2}\int_0^t X_s ds-\frac{\mu}{(logt)^2}\int_0^t\frac{ds}{X_s}}\right) &= \lim_{t\to\infty}\frac{\exp\left(\frac{\sqrt{\mu}}{\sqrt{\sigma}\log t}\log\left(\frac{\sqrt{\sigma\lambda}x}{t\sigma\sinh(\sqrt{\sigma\lambda})}\right)\right)}{(\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda})+\cosh(\sqrt{\sigma\lambda})} \\
&= \frac{\exp\left(\frac{\sqrt{\mu}}{\sqrt{\sigma}}\right)}{(\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda})+\cosh(\sqrt{\sigma\lambda})} \\
&= \mathbb{E}_x\left(e^{-\rho R_1-\lambda I_1}\right)\mathbb{E}_x\left(e^{-\mu\tau_1}\right).
\end{aligned}$$

Which completes the proof. □

We now turn to the case $b \neq 0$.

2.2 Case $b \neq 0$

Let us resume the general model of the CIR given by relation (1) with $b \neq 0$, namely

$$dX_t = (a-bX_t)dt+\sqrt{2\sigma X_t}dW_t, \qquad (8)$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}^*$, $\sigma > 0$. Note that this process may be represented in terms of a square Bessel process through the relation $X_t = e^{-bt}Y\left(\frac{\sigma}{2b}(e^{bt}-1)\right)$, where Y denotes a $\mathrm{BESQ}_x^{\frac{2a}{\sigma}}$. This relation results from simple properties of square Bessel processes (see e.g. Göing-Jaeschke and Yor [6] and Revuz and Yor [16]). In the same manner as for the case $b = 0$, Ben Alaya and Kebaier study the seperate asymptotic behavior of $(X_t, \int_0^t X_s ds)$ and $\int_0^t \frac{ds}{X_s}$ (see propositions 3 and 4 of [1]) and prove the following result

Theorem 3 *Let $(X_t)_{t\geq 0}$ be a CIR process solution to (8), we have*

1. *If $b > 0$ then $\frac{1}{t}\int_0^t X_s ds \xrightarrow{\mathbb{P}} \frac{a}{b}$ as t tends to infinity.*

2. *If $b < 0$ then $\left(e^{bt}X_t, e^{bt}\int_0^t X_s ds\right) \xrightarrow{law} (R_{t_0}, t_0 R_{t_0})$, as t tends to infinity, where $t_0 = -1/b$ and $(R_t)_{t\geq 0}$ is the CIR process, starting from x, solution to (2).*

3. $\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = 1$ *if and only if $a \geq \sigma$.*

4. *If $b > 0$ and $a > \sigma$ then $\frac{1}{t}\int_0^t \frac{ds}{X_s} \xrightarrow{\mathbb{P}} \frac{b}{a-\sigma}$ as t tends to infinity.*

5. *If $b > 0$ and $a = \sigma$ then $\frac{1}{t^2}\int_0^t \frac{ds}{X_s} \xrightarrow{law} \tau_2$ as t tends to infinity, where τ_2 is the hitting time associated with Brownian motion $\tau_2 := \inf\{t > 0 : W_t = \frac{b}{\sigma\sqrt{2}}\}$.*

6. *If $b < 0$ and $a \geq \sigma$ then $\int_0^t \frac{ds}{X_s} \xrightarrow{law} I_{t_0} := \int_0^{t_0} R_s ds$ as t tends to infinity, where $t_0 = -1/b$ and $(R_t)_{t\geq 0}$ is the CIR process, starting from x, solution to (2).*

According to the third assertion of the above theorem our next result makes sense if and only if $a \geq \sigma$ and we have.

Theorem 4 *Let $(X_t)_{t\geq 0}$ be a CIR process solution to (8), we have*

1. *If $b > 0$ and $a > \sigma$ then $\left(\frac{1}{t}\int_0^t X_s ds, \frac{1}{t}\int_0^t \frac{ds}{X_s}\right) \xrightarrow{\mathbb{P}} \left(\frac{a}{b}, \frac{b}{a-\sigma}\right)$ as t tends to infinity.*

2. *If $b > 0$ and $a = \sigma$ then $\left(\frac{1}{t}\int_0^t X_s ds, \frac{1}{t^2}\int_0^t \frac{ds}{X_s}\right) \xrightarrow{law} \left(\frac{a}{b}, \tau_2\right)$ as t tends to infinity, where τ_2 is the hitting time associated with Brownian motion $\tau_2 := \inf\{t > 0 : W_t = \frac{b}{\sqrt{2\sigma}}\}$.*

3. *If $b < 0$ and $a \geq \sigma$ then $\left(e^{bt}X_t, e^{bt}\int_0^t X_s ds, \int_0^t \frac{ds}{X_s}\right) \xrightarrow{law} (R_{t_0}, t_0 R_{t_0}, I_{t_0})$, as t tends to infinity, where $t_0 = -1/b$, $(R_t)_{t\geq 0}$ is the CIR process, starting from x solution to (2), and $I_{t_0} := \int_0^{t_0} R_s ds$.*

Proof of Theorem The two first assertions are straightforward consequence from Theorem 3. For the case $b<0$ and $a\geq\sigma$, we have only to note that $(\int_0^t \frac{ds}{X_s})_{t\geq 0}$ is an increasing process converging to a random variable with the same law as I_{t_0}. The result follows by assertion 2 of Proposition 3 of [1]. $\square$

Nevertheless, the above theorem is not sufficient to prove all the asymptotic results we manage to establish in our next section. Therefore, we need to prove in extra the following result.

Proposition 2 *For $\rho\geq 0, \lambda\geq 0$ and $\mu>0$, we have*

$$
\begin{aligned}
\mathbb{E}_x\left(e^{-\rho X_t-\lambda\int_0^t X_s ds-\mu\int_0^t \frac{ds}{X_s}}\right) &= \frac{\Gamma(k+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\nu+1)}\exp\left(\frac{b}{2\sigma}(at+x)-\frac{Ax}{2\sigma}\coth(At/2)\right) \\
&\times\left(\frac{Ax}{2\sigma\sinh(At/2)}\right)^{\frac{\nu}{2}+\frac{1}{2}-k}\left(\frac{2\sigma\rho+b}{A}\sinh(At/2)+\cosh(At/2)\right)^{-\frac{\nu}{2}-\frac{1}{2}-k} \\
&\times\ {}_1F_1\left(k+\frac{\nu}{2}+\frac{1}{2},\nu+1,\frac{A^2x}{2\sigma\sinh(At/2)\left((2\sigma\rho+b)\sinh(At/2)+A\cosh(At/2)\right)}\right)
\end{aligned} \tag{9}
$$

where $k=\frac{a}{2\sigma}$, $A=\sqrt{b^2+4\sigma\lambda}$ and $\nu=\frac{1}{\sigma}\sqrt{(a-\sigma)^2+4\mu\sigma}$.

Proof We apply Theorem 5.10 of [3], for $b\neq 0$. We obtain the fundamental solution of the PDE $u_t=\sigma x u_{xx}+(a-bx)u_x-(\lambda x+\frac{\mu}{x})u$, $\mu>0$ and $\lambda>0$:

$$
\begin{aligned}
p(t,x,y) = & \frac{A}{2\sigma\sinh(At/2)}\left(\frac{y}{x}\right)^{a/(2\sigma)-1/2} \\
& \times\exp\left(\frac{b}{2\sigma}\left[at+(x-y)\right]-\frac{A(x+y)}{2\sigma\tanh(At/2)}\right)I_\nu\left(\frac{A\sqrt{xy}}{\sigma\sinh(At/2)}\right).
\end{aligned} \tag{10}
$$

Since $\mathbb{E}_x\left(e^{-\rho X_t-\lambda\int_0^t X_s ds-\mu\int_0^t \frac{ds}{X_s}}\right)=\int_0^\infty e^{-\rho y}p(t,x,y)dy$, we deduce the Laplace transform of the triplet $(X_t,\int_0^t X_s ds,\int_0^t \frac{ds}{X_s})$. In the same manner as in the proof of Proposition 3, formula 2 of section 6.643 in [7] gives us

$$
\begin{aligned}
\mathbb{E}_x\left(e^{-\rho X_t-\lambda\int_0^t X_s ds-\mu\int_0^t \frac{ds}{X_s}}\right) &= \frac{\Gamma(k+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\nu+1)}x^{-k}\exp\left(\frac{b}{2\sigma}(at+x)-\frac{Ax}{2\sigma}\coth(At/2)\right) \\
&\times\left(\frac{2\sigma\sinh(At/2)}{(2\sigma\rho+b)\sinh(At/2)+A\cosh(At/2)}\right)^k \\
&\times\exp\left(\frac{A^2x}{4\sigma\sinh(At/2)\left((2\sigma\rho+b)\sinh(At/2)+A\cosh(At/2)\right)}\right) \\
&\times M_{-k,\frac{\nu}{2}}\left(\frac{A^2x}{2\sigma\sinh(At/2)\left((2\sigma\rho+b)\sinh(At/2)+A\cosh(At/2)\right)}\right).
\end{aligned} \tag{11}
$$

Finally, by inserting relation (7) in (11) we obtain the announced result.

$\square$

3 Estimation of the CIR diffusion from continuous observations

Let us first recall some basic notions on the construction of the maximum likelihood estimator (MLE). Suppose that the one dimensional diffusion process $(X_t)_{t\geq 0}$ satisfies

$$dX_t = b(\theta, X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0,$$

where the parameter $\theta \in \Theta \subset \mathbb{R}^p$, $p \geq 1$, is to be estimated. The coefficients b and σ are two functions satisfying conditions that guarantee the existence and uniqueness of the SDE for each $\theta \in \Theta$. We denote by $\mathbb{P}_\theta$ the probability measure induced by the solution of the equation on the canonical space $C(\mathbb{R}_+, \mathbb{R})$ with the natural filtration $\mathcal{F}_t := \sigma(W_s, s \leq t)$, and let $\mathbb{P}_{\theta,t} := \mathbb{P}_\theta | \mathcal{F}_t$ be the restriction of $\mathbb{P}_\theta$ to $\mathcal{F}_t$. If the integrals in the next formula below make sense then the measures $\mathbb{P}_{\theta,t}$ and $\mathbb{P}_{\theta_0,t}$, for any $\theta, \theta_0 \in \Theta, t > 0$, are equivalent (see Jacod [9] and Lipster and Shirayev [13]) and we are able to introduce the so called likelihood ratio

$$L_t^{\theta,\theta_0} := \frac{d\mathbb{P}_{\theta,t}}{d\mathbb{P}_{\theta_0,t}} = \exp\left\{ \int_0^t \frac{b(\theta, X_s) - b(\theta_0, X_s)}{\sigma^2(X_s)} dX_s - \frac{1}{2} \int_0^t \frac{b^2(\theta, X_s) - b^2(\theta_0, X_s)}{\sigma^2(X_s)} ds \right\}. \quad (12)$$

The process $\left(L_t^{\theta,\theta_0}\right)_{t\geq 0}$ is an $\mathcal{F}_t-$martingale.

In the present section, we observe the process $X^T = (X_t)_{0\leq t\leq T}$ as a parametric model solution to equation (1), namely

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t} dW_t,$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$. The unknown parameter, say θ, is involved only in the drift part of the diffusion and we consider the case $\theta = (a, b)$.

3.1 Parameter estimation $\theta = (a, b)$

According to relation (12), the appropriate likelihood ratio, evaluated at time T with $\theta_0 = (0, 0)$, makes sense when $\mathbb{P}_\theta(\int_0^T \frac{ds}{X_s} < \infty) = 1$ and is given by

$$L_T(\theta) = L_T(a, b) := L_T^{\theta,\theta_0} = \exp\left\{ \frac{1}{2\sigma} \int_0^T \frac{a - bX_s}{X_s} dX_s - \frac{1}{4\sigma} \int_0^T \frac{(a - bX_s)^2}{X_s} ds \right\}.$$

Hence, for $a \geq \sigma$, the MLE $\hat{\theta}_T = (\hat{a}_T, \hat{b}_T)$ of $\theta = (a, b)$ that maximizes $L_T(a, b)$ is well defined and we have

$$\begin{cases} \hat{a}_T &= \dfrac{\int_0^T X_s ds \int_0^T \frac{dX_s}{X_s} - T(X_T - x)}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \\ \\ \hat{b}_T &= \dfrac{T \int_0^T \frac{dX_s}{X_s} - (X_T - x) \int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \end{cases}$$

Hence, the error is given by

$$\begin{cases} \hat{a}_T - a &= \sqrt{2\sigma}\dfrac{\int_0^T X_s ds \int_0^T \frac{dW_s}{\sqrt{X_s}} - T\int_0^T \sqrt{X_s} dW_s}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \\ \\ \hat{b}_T - b &= \sqrt{2\sigma}\dfrac{T\int_0^T \frac{dW_s}{\sqrt{X_s}} - \int_0^T \frac{ds}{X_s}\int_0^T \sqrt{X_s} dW_s}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \end{cases}$$

The above error is obviously of the form

$$\hat{\theta}_T - \theta = \sqrt{2\sigma}\langle M\rangle_T^{-1} M_T, \text{ with } M_T = \begin{pmatrix} \int_0^T \frac{dW_s}{\sqrt{X_s}} \\ \\ -\int_0^T \sqrt{X_s} dW_s \end{pmatrix} \tag{13}$$

and $(\langle M\rangle_t)_{t\geq 0}$ is the quadratic variation of the Brownian martingale $(M_t)_{t\geq 0}$. If this quadratic variation, correctly normalized, converges in probability then the classical martingale central limit theorem can be applied. Here and in the following $\Longrightarrow$ means the convergence in distribution under $\mathbb{P}_\theta$.

Theorem 5 *For the case $b > 0$ and $a > \sigma$*

$$\mathcal{L}_\theta\left\{\sqrt{T}(\hat{\theta}_T - \theta)\right\} \Longrightarrow \mathcal{N}_{\mathbb{R}^2}\left(0, 2\sigma C^{-1}\right), \text{ with } C = \begin{pmatrix} \frac{b}{a-\sigma} & -1 \\ -1 & \frac{a}{b} \end{pmatrix}.$$

Proof : Since for $a > \sigma$ and $b > 0$ the CIR process is ergodic and the stationary distribution is a Gamma law with shape a/σ and scale σ/b, the ergodic theorem yields

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_s ds = \frac{a}{b}, \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{ds}{X_s} = \frac{b}{a-\sigma} \text{ and } \lim_{T\to\infty}\frac{\langle M\rangle_T}{T} = C \quad \mathbb{P}_\theta \ a.s..$$

Therefore, by the martingale central limit theorem we get $\mathcal{L}_\theta\left\{\frac{1}{\sqrt{T}}M_T\right\} \Longrightarrow \mathcal{N}_{\mathbb{R}^2}(0, C)$. We complete the proof using identity (13).

□

In the following, it is relevant to rewrite, using Itô's formula, the MLE error as follows

$$\begin{cases} \hat{a}_T - a &= \dfrac{\left(\log X_T - \log x + (\sigma - a)\int_0^T \frac{ds}{X_s}\right)\int_0^T X_s ds - T\left(X_T - x - aT\right)}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \\ \\ \hat{b}_T - b &= \dfrac{T\left(\log X_T - \log x + bT + \sigma\int_0^T \frac{ds}{X_s}\right) - \left(X_T - x + b\int_0^T X_s ds\right)\int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2}. \end{cases} \tag{14}$$

Consequently, the study of the vector $(X_T, \int_0^T X_s, \int_0^T \frac{ds}{X_s})$, done in the previous section, will be very helpful to investigate the limit law of the error. The asymptotic behavior of $\hat{\theta}_T - \theta$ can be summarized as follows.

Theorem 6 *The MLE of $\theta = (a, b)$ is well defined for $a \geq \sigma$ and satisfies*

1. *Case $b = 0$ and $a = \sigma$: $\mathcal{L}_\theta \left\{ \mathrm{diag}(\log T, T)(\hat{\theta}_T - \theta) \right\} \Longrightarrow \left(\frac{1}{\tau_1}, \frac{a - R_1}{I_1} \right)$, where (R_t) is the CIR process, starting from 0, solution to (2), $I_t = \int_0^t R_s ds$, and τ_1 is the hitting time associated with Brownian motion $\tau_1 = \inf\{t > 0 : W_t = \frac{1}{\sqrt{2\sigma}}\}$. The couple (R_1, I_1) and the random time τ_1 are independent.*

2. *Case $b = 0$ and $a > \sigma$: $\mathcal{L}_\theta \left\{ \mathrm{diag}(\sqrt{\log T}, T)(\hat{\theta}_T - \theta) \right\} \Longrightarrow \left(\sqrt{2\sigma(a - \sigma)} G, \frac{a - R_1}{I_1} \right)$, where (R_1, I_1) is defined in the previous case, G is a standard normal random variable indepedent of (R_1, I_1).*

Proof : For the case $b = 0$ and $a = \sigma$

$$\mathrm{diag}(\log T, T)(\hat{\theta}_T - \theta) = \begin{cases} \dfrac{\frac{\log X_T - \log x}{\log T} \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{X_T - x - aT}{T \log T}}{\left(\frac{1}{(\log T)^2} \int_0^T \frac{ds}{X_s} \right) \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{1}{(\log T)^2}} \\[2ex] \dfrac{\frac{\log X_T - \log x}{(\log T)^2} + \frac{a}{(\log T)^2} \int_0^T \frac{ds}{X_s} - \frac{X_T - x}{T} \times \left(\frac{1}{(\log T)^2} \int_0^T \frac{ds}{X_s} \right)}{\left(\frac{1}{(\log T)^2} \int_0^T \frac{ds}{X_s} \right) \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{1}{(\log T)^2}} \end{cases}$$

By a scaling argument the process $(X_{2t/a})$ has the same distribution as a bidimensional square Bessel process, starting from x, BESQ_x^2. It follows that

$$X_{2T/a} \overset{law}{=} \|B_T + x\|^2 \overset{law}{=} T \|B_1 + x/\sqrt{T}\|^2,$$

where $(B_t)_{t \geq 0}$ denotes a standard bidimensional Brownian motion. Hence, $\log X_T / \log T$ converges in law to one and consequently in probability. We complete the proof of the first case using the second assertion of Theorem 2. For the case $b = 0$ and $a > \sigma$, we have

$$\mathrm{diag}(\sqrt{\log T}, T)(\hat{\theta}_T - \theta) = \begin{cases} \dfrac{\frac{\log X_T - \log x + (\sigma - a) \int_0^T \frac{ds}{X_s}}{\sqrt{\log T}} \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{X_T - x - aT}{T \sqrt{\log T}}}{\left(\frac{1}{\log T} \int_0^T \frac{ds}{X_s} \right) \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{1}{\log T}} \\[2ex] \dfrac{\frac{\log X_T - \log x + \sigma \int_0^T \frac{ds}{X_s}}{\log T} - \frac{X_T - x}{T} \times \left(\frac{1}{\log T} \int_0^T \frac{ds}{X_s} \right)}{\left(\frac{1}{\log T} \int_0^T \frac{ds}{X_s} \right) \times \left(\frac{1}{T^2} \int_0^T X_s ds \right) - \frac{1}{\log T}} \end{cases}.$$

Note that for any $u \in \mathbb{R}$, $v > 0$ and $w > 0$ we have

$$\mathbb{E}_x \left(\exp \left[u \frac{\log X_T + (\sigma - a) \int_0^T \frac{ds}{X_s}}{\sqrt{\log T}} - \frac{v}{T^2} \int_0^T X_s ds - \frac{w}{T} X_T \right] \right) =$$

$$\mathbb{E}_x \left(X_t^\eta \exp \left[-\rho X_T - \lambda \int_0^T X_s ds - \mu \int_0^T \frac{ds}{X_s} \right] \right)$$

with $\eta = \frac{u}{\sqrt{\log T}}$, $\rho = \frac{w}{T}$, $\lambda = \frac{v}{T^2}$ and $\mu = \frac{u(a-\sigma)}{\sqrt{\log T}}$. The moment generating-Laplace transform in the right hand side of the above equality is given by relation (3) of Proposition 1 . Consequently, using standard evaluations, it is easy to prove that the limit of the last quantity, when T tends to infinity, is equal to

$$\left(\frac{\sqrt{\sigma v}}{\sigma w \sinh(\sqrt{\sigma v}) + \sqrt{\sigma \mu}\cosh(\sqrt{\sigma v})}\right)^{\frac{a}{\sigma}} \times$$

$$\lim_{T\to+\infty} \exp\left[-\log T\left(\frac{1}{2\sigma}\sqrt{(a-\sigma)^2 + \frac{4u\sigma(a-\sigma)}{\sqrt{\log T}}} + \frac{\sigma - a}{2\sigma} - \frac{u}{\sqrt{\log T}}\right)\right]$$

$$= \left(\frac{\sqrt{\sigma v}}{\sigma w \sinh(\sqrt{\sigma v}) + \sqrt{\sigma \mu}\cosh(\sqrt{\sigma v})}\right)^{\frac{a}{\sigma}} \exp\left(\frac{\sigma}{a-\sigma}u^2\right).$$

It follows that

$$\left(\frac{\log X_T + (\sigma - a)\int_0^T \frac{ds}{X_s}}{\sqrt{\log T}}, \frac{1}{T^2}\int_0^T X_s ds, \frac{X_T}{T}\right) \xrightarrow{law} \left(\sqrt{\frac{2\sigma}{a-\sigma}}G, I_1, R_1\right) \text{ as } T \text{ tends to infinity.}$$

We complete the proof of the second assertion using that $\frac{1}{\log T}\int_0^T \frac{ds}{X_s} \xrightarrow{\mathbb{P}} \frac{1}{a-\sigma}$ (see the first assertion of Theorem 2) and that $\frac{\log X_T}{\log T} \xrightarrow{\mathbb{P}} 1$ which is a starightforward consequence of the above result, namely: $\frac{X_T}{T} \xrightarrow{law} R_1$. □

Theorem 7 *The MLE of $\theta = (a, b)$ is well defined for $a \geq \sigma$ and satisfies*

1. *Case $b > 0$ and $a = \sigma$: $\mathcal{L}_\theta\left\{\operatorname{diag}(T, \sqrt{T})(\hat{\theta}_T - \theta)\right\} \Longrightarrow \left(\frac{b}{\tau_2}, \sqrt{2b}G\right)$, where G is a standard normal random variable indepedent of τ_2 the hitting time associated with Brownian motion $\tau_2 = \inf\{t > 0 : W_t = \frac{b}{\sqrt{2\sigma}}\}$.*

2. *Case $b < 0$ and $a \geq \sigma$: the MLE estimator $\hat{\theta}_T$ is not consistent.*

Proof : For the first case we have

$$\operatorname{diag}(T, \sqrt{T})(\hat{\theta}_T - \theta) = \begin{cases} \dfrac{\frac{\log X_T - \log x}{T} \times \left(\frac{1}{T}\int_0^T X_s ds\right) - \frac{X_T - x - aT}{T}}{\left(\frac{1}{T^2}\int_0^T \frac{ds}{X_s}\right) \times \left(\frac{1}{T}\int_0^T X_s ds\right) - \frac{1}{T}} \\[2em] \dfrac{\frac{\log X_T - \log x + bT}{T\sqrt{T}} - \frac{X_T - x - aT + b\int_0^T X_s ds}{\sqrt{T}} \times \left(\frac{1}{T^2}\int_0^T \frac{ds}{X_s}\right)}{\left(\frac{1}{T^2}\int_0^T \frac{ds}{X_s}\right) \times \left(\frac{1}{T}\int_0^T X_s ds\right) - \frac{1}{T}} \end{cases}$$

At first, note that the term $\frac{X_T}{T}$ appearing in the first component of the above relation vanishes as T tends to infinity. This is straightforward by combining that

$$\frac{X_T}{T} = \frac{x+a}{T} - \frac{b}{T}\int_0^T X_s\,ds + \frac{\sqrt{2a}}{T}\int_0^T \sqrt{X_s}\,dW_s,$$

with the classical large law number property for continuous martingales which applies here, since $\frac{1}{T}\int_0^T X_s ds \xrightarrow{\mathbb{P}} \frac{a}{b}$ (see assertion 2 of Theorem 4). Now, note that for any $u \in \mathbb{R}$ and $v > 0$, we have

$$\mathbb{E}_x\left(\exp\left[u\frac{X_T - aT + b\int_0^T X_s ds}{\sqrt{T}} - \frac{v}{T^2}\int_0^T \frac{ds}{X_s}\right]\right)$$
$$= e^{-au\sqrt{T}}\mathbb{E}_x\left(\exp\left[-\rho X_T - \lambda\int_0^T X_s ds - \mu\int_0^T \frac{ds}{X_s}\right]\right)$$

with $\rho = -\frac{u}{\sqrt{T}}$, $\lambda = -\frac{ub}{\sqrt{T}}$ and $\mu = \frac{v}{T^2}$. The moment generating-Laplace transform in the right hand side of the above equality is given by relation (9) of Proposition 2 with $a = \sigma$. Using standard evaluations, it is easy to prove that the limit of the last quantity, when T tends to infinity, is equal to

$$\lim_{T\to\infty} \frac{1}{2}\exp\left(\frac{bT}{2} - au\sqrt{T}\right)\left(\sinh\left(\frac{T}{2}\sqrt{b^2 - \frac{4aub}{\sqrt{T}}}\right)\right)^{-\frac{2\sqrt{v}}{T\sqrt{\sigma}}-1}$$
$$= \lim_{T\to\infty} \exp\left(\frac{bT}{2} - au\sqrt{T}\right)\exp\left((-\frac{2\sqrt{v}}{T\sqrt{\sigma}} - 1)\left(\frac{T}{2}\sqrt{b^2 - \frac{4aub}{\sqrt{T}}}\right)\right)$$
$$= \exp\left(\frac{\sigma^2 u^2}{b} - \frac{b\sqrt{u}}{\sqrt{\sigma}}\right)$$

Hence the couple $\left(\frac{X_T - aT + b\int_0^T X_s ds}{\sqrt{T}}, \frac{1}{T^2}\int_0^T \frac{ds}{X_s}\right)$ converges in law to $\left(\sqrt{\frac{2}{b}}\sigma G, \tau_2\right)$ when T tends to infinity. On the other hand, using that the CIR process (X_t) can be represented in terms of a BESQ_x^2 as follows

$$X_T \overset{law}{=} e^{-bT}\mathrm{BESQ}_x^2\left(\frac{a}{2b}(e^{bT} - 1)\right)$$

and that $\log \mathrm{BESQ}_x^2(T)/\log T$ converges in probability to one, we obtain that $(\log X_T - \log x + bT)/T$ converges in probability to b and consequently we deduce $\dfrac{\log X_T}{T} \xrightarrow{\mathbb{P}} 0$. We complete the proof of the first assertion using that $\dfrac{1}{T}\displaystyle\int_0^T X_s ds \xrightarrow{\mathbb{P}} \dfrac{a}{b}$ (see assertion 2 of Theorem 4).

For the second case, $b < 0$ and $a \geq \sigma$, we have

$$\begin{aligned}
\hat{a}_T - a &= \sqrt{2\sigma}\frac{\int_0^T X_s ds \int_0^T \frac{dW_s}{\sqrt{X_s}} - T\int_0^T \sqrt{X_s} dW_s}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \\
&= \sqrt{2\sigma}\frac{\int_0^T \frac{dW_s}{\sqrt{X_s}} - \dfrac{Te^{bT}\int_0^T \sqrt{X_s} dW_s}{e^{bT}\int_0^T X_s ds}}{\int_0^T \frac{ds}{X_s} - \dfrac{T^2 e^{bT}}{e^{bT}\int_0^T X_s ds}}.
\end{aligned}$$

Since $\left(\int_0^t \frac{ds}{X_s}\right)_{t\geq 0}$ is an increasing process converging to a finite random variable, we easily deduce the almost sure convergence of the Brownian martingale $\left(\int_0^t \frac{dW_s}{\sqrt{X_s}}\right)_{t\geq 0}$. Thanks to the convergence in law of the term $e^{bT}\int_0^T X_s ds$, we finish the proof using that

$$T^2 e^{2bT}\mathbb{E}\left(\int_0^T \sqrt{X_s} dW_s\right)^2 = T^2 e^{2bT}\int_0^T \mathbb{E}(X_s) ds = T^2 e^{2bT}\int_0^T \left(\frac{a}{b} + (x - \frac{a}{b})e^{-bs}\right) ds \underset{T\to\infty}{\longrightarrow} 0.$$

□

4 Estimation of the CIR diffusion from discrete observations

In this section, we consider rather a discrete sample $(X_{t_k})_{0\leq k\leq n}$ of the CIR diffusion at deterministic and equidistant instants $(t_k = k\Delta_n)_{0\leq k\leq n}$. Our aim is to study a new estimator for $\theta = (a, b)$ based on discrete observations, under the conditions of high frequency, $\Delta_n \to 0$, and infinite horizon, $n\Delta_n \to \infty$. A common way to do that is to consider a discretization of the logarithm likelihood (see [10] and references there). In our case this method yields the contrast

$$\frac{1}{2\sigma}\sum_{k=0}^{n-1}\frac{a - bX_{t_k}}{X_{t_k}}(X_{t_{k+1}} - X_{t_k}) - \frac{1}{4\sigma}\sum_{k=0}^{n-1}\Delta_n\frac{(a - bX_{t_k})^2}{X_{t_k}},$$

Our approach is slightly different since we discretize the continuous time MLE, obtained in the previous section, instead of considering the maximum argument of the above contrast. Doing so, we take advantage from limit theorems obtained in the continuous time observations. Hence, thanks to relation (14), the discretized version of the MLE is given by

$$\begin{cases}
\hat{a}_{t_n}^{\Delta_n} &= \dfrac{\left(\log X_{t_n} - \log x + \sigma\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right)\sum_{k=0}^{n-1}\Delta_n X_{t_k} - t_n\left(X_{t_n} - x\right)}{\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\sum_{k=0}^{n-1}\Delta_n X_{t_k} - t_n^2} \\
\hat{b}_{t_n}^{\Delta_n} &= \dfrac{t_n\left(\log X_{t_n} - \log x + \sigma\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right) - (X_{t_n} - x)\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}}{\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\sum_{k=0}^{n-1}\Delta_n X_{t_k} - t_n^2}.
\end{cases} \tag{15}$$

In order to prove limit theorems on the discrete estimator, $\hat{\theta}^{\Delta_n}_{t_n} := (\hat{a}^{\Delta_n}_{t_n}, \hat{b}^{\Delta_n}_{t_n})$, we need to control the errors $\int_0^{t_n} X_s ds - \sum_{k=0}^{n-1} \Delta_n X_{t_k}$ and $\int_0^{t_n} \frac{ds}{X_s} - \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}$ and somme moments behavior on the increments of the CIR process are needed.

4.1 Moment properties of the CIR process

Let us first, yield some essential properties on moments of the CIR process.

Proposition 3 *For all $\eta \in]-\frac{a}{\sigma}, +\infty[$ we have*

1. *For $b = 0$, we have*
$$\mathbb{E}_x(X_t^\eta) \sim \sigma^\eta \frac{\Gamma(\frac{a}{\sigma}+\eta)}{\Gamma(\frac{a}{\sigma})} t^\eta, \text{ as } t \to +\infty,$$
and we have, $\sup_{0\leq t\leq 1} \mathbb{E}_x(X_t^\eta) < \infty$ *and* $\sup_{t\geq 1} \dfrac{\mathbb{E}_x(X_t^\eta)}{t^\eta} < \infty$.

2. *For $b > 0$, we have*
$$\mathbb{E}_x(X_t^\eta) \sim \left(\frac{\sigma^2}{2b}\right)^\eta \frac{\Gamma(\frac{a}{\sigma}+\eta)}{\Gamma(\frac{a}{\sigma})}, \text{ as } t \to +\infty,$$
and we have, $\sup_{t\geq 0} \mathbb{E}_x(X_t^\eta) < \infty$.

Proof : For the first assertion we take $\rho = \lambda = 0$ and let μ tend to 0 in relation (3) of Proposition 1, it follows that

$$\mathbb{E}_x(X_t^\eta) = (\sigma t)^\eta \frac{\Gamma(\frac{a}{\sigma}+\eta)}{\Gamma(\frac{a}{\sigma})} \exp\left(-\frac{x}{\sigma t}\right) {}_1F_1\left(\frac{a}{\sigma}+\eta, \frac{a}{\sigma}, \frac{x}{\sigma t}\right)$$

and we get the result using that

$$\lim_{t\to\infty} {}_1F_1\left(\frac{a}{\sigma}+\eta, \frac{a}{\sigma}, \frac{x}{\sigma t}\right) = 1.$$

The second assertion is immediate using the previous assertion and that, for $b > 0$, we have the relation $X_t = e^{-bt} Y\left(\frac{\sigma}{2b}(e^{bt}-1)\right)$, where Y denotes a $\mathrm{BESQ}_x^{\frac{2a}{\sigma}}$. As the function $t \mapsto \mathbb{E}_x(X_t^\eta)$ is continuous we obtain the last result. □

In the following, C denotes a positive constant that may change from line to line.

Proposition 4 *In the case $b > 0$, let $0 \leq s < t$ such that $0 < t - s < 1$ we have*

1. *For all $q \geq 1$,*
$$\mathbb{E}_x|X_t - X_s|^q \leq C(t-s)^{\frac{q}{2}}.$$

2. *For all $a > 2\sigma$,*
$$\mathbb{E}_x\left|\frac{1}{X_t} - \frac{1}{X_s}\right| \leq C(t-s)^{\frac{1}{2}}.$$

Proof : First, note that by linearization technics we obtain

$$d(e^{bt}X_t) = ae^{bt}dt + e^{bt}\sqrt{2\sigma X_t}dW_t. \tag{16}$$

Hence,

$$X_t - X_s = \left(\frac{a}{b} - X_s\right)\left(1 - e^{-b(t-s)}\right) + \int_s^t e^{-b(t-u)}\sqrt{2\sigma X_u}dW_u.$$

Let $q \geq 1$,

$$\mathbb{E}_x|X_t - X_s|^q \leq 2^{q-1}\left(1 - e^{-b(t-s)}\right)^q \mathbb{E}_x\left|\frac{a}{b} - X_s\right|^q + 2^{q-1}\mathbb{E}_x\left|\int_s^t e^{-b(t-u)}\sqrt{2\sigma X_u}dW_u\right|^q.$$

The first term in the right hand side is bounded by $C(t-s)^q$ since $1 - e^{-x} \leq x$, for $x \geq 0$, and $\sup_{t\geq 0}\mathbb{E}_x(X_t^q) < \infty$ (see proposition 3). For the second term, the Burkholder-Davis-Gundy inequality yields

$$\mathbb{E}_x\left(\int_s^t e^{-b(t-u)}\sqrt{2\sigma X_u}dW_u\right)^q \leq \mathbb{E}_x\left(\int_s^t 2\sigma e^{-2b(t-u)}X_u du\right)^{\frac{q}{2}}$$

For $q \geq 2$, we apply the Holder inequality on the integral to obtain

$$\begin{aligned}\mathbb{E}_x\left(\int_s^t e^{-b(t-u)}\sqrt{2\sigma X_u}dW_u\right)^q &\leq C(t-s)^{\frac{q}{2}-1}\int_s^t e^{-bq(t-u)}\mathbb{E}_x(X_u^{\frac{q}{2}})\,du \\ &\leq C(t-s)^{\frac{q}{2}},\end{aligned}$$

since the integrand is bounded using $\sup_{u\geq 0}\mathbb{E}_x(X_u^{\frac{q}{2}}) < \infty$. For $1 \leq q < 2$, we apply the Holder inequality on the expectation and we obtain

$$\begin{aligned}\mathbb{E}_x\left(\int_s^t e^{-b(t-u)}\sqrt{2\sigma X_u}dW_u\right)^q &\leq \left(\int_s^t 2\sigma e^{-2b(t-u)}\mathbb{E}_x(X_u)du\right)^{\frac{q}{2}} \\ &\leq C(t-s)^{\frac{q}{2}},\end{aligned}$$

since the integrand is also bounded using once again $\sup_{u\geq 0}\mathbb{E}_x(X_u) < \infty$. Combining these three upper bounds with $0 < t - s < 1$, we deduce the first assertion. Concerning the second assertion, we use the Holder inequality with $\frac{1}{q} + \frac{2}{p} = 1$ and $2 < p < \frac{a}{\sigma}$ which ensures the boundedness of all the terms. We obtain

$$\mathbb{E}_x\left|\frac{1}{X_t} - \frac{1}{X_s}\right| \leq \|X_t - X_s\|_q \left\|X_t^{-1}\right\|_p \left\|X_s^{-1}\right\|_p \leq C(t-s)^{\frac{1}{2}},$$

since by Proposition 3 we have $\sup_{t\geq 0}\mathbb{E}_x(X_t^{-p}) < \infty$, for $p < \frac{a}{\sigma}$, and $\mathbb{E}_x|X_t - X_s|^q \leq C(t-s)^{\frac{q}{2}}$, for $q \geq 2$. □

Proposition 5 *In the case $b = 0$, let $0 \leq s < t$ such that $0 < t - s < 1$ we have*

1. *For all $q \geq 2$,*

$$\mathbb{E}_x|X_t - X_s|^q \leq C(t-s)^{\frac{q}{2}}\sup_{s\leq u\leq t}\mathbb{E}_x(X_u^{\frac{q}{2}}).$$

2. *For all* $1 \leq q < 2$,
$$\mathbb{E}_x|X_t - X_s|^q \leq C(at+x)^{\frac{q}{2}}(t-s)^{\frac{q}{2}}.$$

3. *For all* $a > 2\sigma$, *there exists* $q \geq 2$ *and* $2 < p < \frac{a}{\sigma}$, *such that*
$$\mathbb{E}_x \left| \frac{1}{X_t} - \frac{1}{X_s} \right| \leq C(t-s)^{\frac{1}{2}} \sup_{s \leq u \leq t} \left(\mathbb{E}_x(X_u^{\frac{q}{2}}) \right)^{\frac{1}{q}} \left\| X_t^{-1} \right\|_p \left\| X_s^{-1} \right\|_p.$$

Proof : First, we have
$$X_t - X_s = a(t-s) + \int_s^t \sqrt{2\sigma X_u} dW_u.$$

Let $q \geq 1$,
$$\mathbb{E}_x|X_t - X_s|^q \leq 2^{q-1} \left(a(t-s)\right)^q + 2^{q-1}\mathbb{E}_x \left| \int_s^t \sqrt{2\sigma X_u} dW_u \right|^q.$$

For the second term, the Burkholder-Davis-Gundy inequality yields
$$\mathbb{E}_x \left(\int_s^t \sqrt{2\sigma X_u} dW_u \right)^q \leq \mathbb{E}_x \left(\int_s^t 2\sigma X_u du \right)^{\frac{q}{2}}$$

For $q \geq 2$, we apply the Holder inequality on the integral to obtain
$$\begin{aligned} \mathbb{E}_x \left(\int_s^t \sqrt{2\sigma X_u} dW_u \right)^q &\leq C(t-s)^{\frac{q}{2}-1} \int_s^t \mathbb{E}_x(X_u^{\frac{q}{2}})\, du \\ &\leq C(t-s)^{\frac{q}{2}} \sup_{s \leq u \leq t} \mathbb{E}_x(X_u^{\frac{q}{2}}), \end{aligned}$$

which completes the second assertion. For $1 \leq q < 2$, we apply the Holder inequality on the expectation and we obtain
$$\begin{aligned} \mathbb{E}_x \left(\int_s^t \sqrt{2\sigma X_u} dW_u \right)^q &\leq \left(\int_s^t 2\sigma \mathbb{E}_x(X_u) du \right)^{\frac{q}{2}} \\ &\leq C(at+x)^{\frac{q}{2}}(t-s)^{\frac{q}{2}}, \end{aligned}$$

using $\mathbb{E}_x(X_u) = au + x$. This completes the second assertion. Concerning the last one, we use the Holder inequality with $\frac{1}{q} + \frac{2}{p} = 1$ and $2 < p < \frac{a}{\sigma}$ which ensures the boundedness of all the terms. We obtain
$$\begin{aligned} \mathbb{E}_x \left| \frac{1}{X_t} - \frac{1}{X_s} \right| &\leq \left\| X_t - X_s \right\|_q \left\| X_t^{-1} \right\|_p \left\| X_s^{-1} \right\|_p \\ &\leq C(t-s)^{\frac{1}{2}} \sup_{s \leq u \leq t} \left(\mathbb{E}_x(X_u^{\frac{q}{2}}) \right)^{\frac{1}{q}} \left\| X_t^{-1} \right\|_p \left\| X_s^{-1} \right\|_p, \end{aligned}$$

since for $q \geq 2$, $\mathbb{E}_x|X_t - X_s|^q$ is bounded by $C(t-s)^{\frac{q}{2}} \sup_{s \leq u \leq t} \mathbb{E}_x(X_u^{\frac{q}{2}})$. □

4.2 Parameter estimation of $\theta = (a, b)$

The task now is to give sufficient conditions on the frequency Δ_n in order to get the same asymptotic results of Theorem 5 and Theorem 6, when we replace the continuous MLE estimator $\hat{\theta}_{t_n} := (\hat{a}_{t_n}, \hat{b}_{t_n})$ by the discrete one $\hat{\theta}_{t_n}^{\Delta_n} := (\hat{a}_{t_n}^{\Delta_n}, \hat{b}_{t_n}^{\Delta_n})$.

Theorem 8 *Under the above notations, for $b > 0$ and $a > 2\sigma$, if $n\Delta_n^2 \to 0$ then we have*

$$\sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n} X_s ds - \frac{1}{t_n}\sum_{k=0}^{n-1}\Delta_n X_{t_k}\right) \longrightarrow 0 \text{ and } \sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n}\frac{1}{X_s}ds - \frac{1}{t_n}\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right) \longrightarrow 0, \tag{17}$$

in L^1 and in probability. Consequently

$$\mathcal{L}_\theta\left\{\sqrt{t_n}(\hat{\theta}_{t_n}^{\Delta_n} - \theta)\right\} \Longrightarrow \mathcal{N}_{\mathbb{R}^2}\left(0, 2\sigma\Gamma^{-1}\right), \text{ with } \Gamma = \begin{pmatrix} \frac{b}{a-\sigma} & -1 \\ -1 & \frac{a}{b} \end{pmatrix}.$$

Proof : For the first convergence in relation (17), thanks to the first assertion of Proposition 4, we consider the L^1 norm and we write

$$\begin{aligned}
\frac{1}{\sqrt{t_n}}\mathbb{E}_x\left|\int_0^{t_n} X_s ds - \sum_{k=0}^{n-1}\Delta_n X_{t_k}\right| &\le \frac{1}{\sqrt{t_n}}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}_x\left|X_s - X_{t_k}\right| ds \\
&\le \frac{C}{\sqrt{t_n}}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\sqrt{s - t_k}\,ds \\
&\le \frac{C}{\sqrt{t_n}}\sum_{k=0}^{n-1}\Delta_n^{\frac{3}{2}} = C\sqrt{n}\Delta_n \longrightarrow 0,
\end{aligned}$$

since $n\Delta_n^2 \longrightarrow 0$. In the same manner, thanks to the second assertion of Propostion 4, we have

$$\begin{aligned}
\frac{1}{\sqrt{t_n}}\mathbb{E}_x\left|\int_0^{t_n}\frac{ds}{X_s} - \sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right| &\le \frac{1}{\sqrt{t_n}}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}_x\left|\frac{1}{X_s} - \frac{1}{X_{t_k}}\right| ds \\
&\le C\sqrt{n}\Delta_n \longrightarrow 0.
\end{aligned}$$

Finally, as $\sqrt{t_n}(\hat{\theta}_{t_n}^{\Delta_n} - \theta) = \sqrt{t_n}(\hat{\theta}_{t_n}^{\Delta_n} - \hat{\theta}_{t_n}) + \sqrt{t_n}(\hat{\theta}_{t_n} - \theta)$ and thanks to Theorem 5, it is sufficient to prove $\sqrt{t_n}(\hat{\theta}_{t_n}^{\Delta_n} - \hat{\theta}_{t_n}) \longrightarrow 0$ in probability. This last convergence is easily obtained by standard arguments using the first part of the theorem. In fact, for instance to prove the convegence of the first compenent we write

$$\sqrt{t_n}(\hat{a}_{t_n}^{\Delta_n} - \hat{a}_{t_n}) = \frac{\hat{B}_{t_n}\sqrt{t_n}(\hat{A}_{t_n}^{\Delta_n} - \hat{A}_{t_n}) - \hat{A}_{t_n}\sqrt{t_n}(\hat{B}_{t_n}^{\Delta_n} - \hat{B}_{t_n})}{\hat{B}_{t_n}^2 + \hat{B}_{t_n}(\hat{B}_{t_n}^{\Delta_n} - \hat{B}_{t_n})}.$$

where

$$\begin{aligned}
\hat{A}_{t_n} &= \left(\tfrac{\log X_{t_n} - \log x}{t_n} + \tfrac{\sigma}{t_n}\int_0^{t_n}\tfrac{ds}{X_s}\right)\left(\tfrac{1}{t_n}\int_0^{t_n} X_s ds\right) - \tfrac{X_{t_n} - x}{t_n} \\
\hat{B}_{t_n} &= \left(\tfrac{1}{t_n}\int_0^{t_n}\tfrac{ds}{X_s}\right)\left(\tfrac{1}{t_n}\int_0^{t_n} X_s ds\right) - 1,
\end{aligned}$$

and $(\hat{A}_{t_n}^{\Delta_n}, \hat{B}_{t_n}^{\Delta_n})$ is simply the discretization version of the couple $(\hat{A}_{t_n}, \hat{B}_{t_n})$. From the first part we have the convergence in probability of $\sqrt{t_n}(\hat{A}_{t_n}^{\Delta_n} - \hat{A}_{t_n}, \hat{B}_{t_n}^{\Delta_n} - \hat{B}_{t_n})$ towards zero and from Theorem 4 we have the convergence in probabilty of the couple $(\hat{A}_{t_n}, \hat{B}_{t_n})$. Which completes the proof. $\square$

Theorem 9 *Under the above notations, for $b = 0$ and $a > 2\sigma$,*

1. *if $n\Delta_n^2 \to 0$ then we have*

$$t_n \left(\frac{1}{t_n^2} \int_0^{t_n} X_s ds - \frac{1}{t_n^2} \sum_{k=0}^{n-1} \Delta_n X_{t_k} \right) \longrightarrow 0, \quad \text{in } L^1 \text{ and in probability.}$$

2. *if $\dfrac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)} \to 0$ then we have*

$$t_n \left(\frac{1}{\log t_n} \int_0^{t_n} \frac{1}{X_s} ds - \frac{1}{\log t_n} \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}} \right) \longrightarrow 0, \quad \text{in } L^1 \text{ and in probability.}$$

3. *Consequently, if $n\Delta_n^2 \to 0$ and $\dfrac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)} \to 0$ then we have*

$$\mathcal{L}_\theta \left\{ \operatorname{diag}(\sqrt{\log t_n}, t_n)(\hat{\theta}_{t_n}^{\Delta_n} - \theta) \right\} \Longrightarrow \left(\sqrt{2\sigma(a-\sigma)} G, \frac{a - R_1}{I_1} \right),$$

where (R_t) is the CIR process, starting from 0, solution to (2), $I_t = \int_0^t R_s ds$, and G is a standard normal random variable indepedent of (R_1, I_1).

Remarks :

Proof : For the first convergence, thanks to the second assertion of Proposition 5, we consider the L^1 norm and we write

$$\begin{aligned}
\frac{1}{t_n} \mathbb{E}_x \left| \int_0^{t_n} X_s ds - \sum_{k=0}^{n-1} \Delta_n X_{t_k} \right| &\leq \frac{1}{t_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}_x \left| X_s - X_{t_k} \right| ds \\
&\leq \frac{C}{t_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \sqrt{as + x} \sqrt{s - t_k} ds \\
&\leq C \frac{\Delta_n^{\frac{1}{2}}}{t_n} \int_0^{t_n} \sqrt{as + x} ds \\
&\leq C \sqrt{n} \Delta_n \longrightarrow 0,
\end{aligned}$$

since $n\Delta_n^2 \longrightarrow 0$. In the same manner, thanks to the second assertion of Propostion 4, we have

$$\begin{aligned}
\frac{t_n}{\log t_n}\mathbb{E}_x\left|\int_0^{t_n}\frac{ds}{X_s}-\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right| &\leq \frac{t_n}{\log t_n}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}_x\left|\frac{1}{X_s}-\frac{1}{X_{t_k}}\right|ds \\
&\leq C\frac{t_n\sqrt{\Delta_n}}{\log t_n}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\sup_{t_k\leq u\leq s}\left(\mathbb{E}_x(X_u^{\frac{q}{2}})\right)^{\frac{1}{q}}\left\|X_{t_k}^{-1}\right\|_p\left\|X_s^{-1}\right\|_p ds,
\end{aligned}$$

using the third assertion of Proposition 5. Thanks to the first assertion of Proposition 3, in one hand, for $s<1$ the integrand is bounded, and in the other hand, for $s\geq 1$,

$$\sup_{t_k\leq u\leq s}\left(\mathbb{E}_x(X_u^{\frac{q}{2}})\right)^{\frac{1}{q}}\leq Cs^{\frac{1}{2}},\quad \left\|X_s^{-1}\right\|_p\leq Cs^{-1},\quad \text{and}\quad \left\|X_{t_k}^{-1}\right\|_p\leq Ct_k^{-1}\leq 2Cs^{-1}\quad \text{when } \Delta_n\leq\frac{1}{2}.$$

It follows that for $s\geq 1$, $\sup_{t_k\leq u\leq s}\left(\mathbb{E}_x(X_u^{\frac{q}{2}})\right)^{\frac{1}{q}}\left\|X_{t_k}^{-1}\right\|_p\left\|X_s^{-1}\right\|_p\leq Cs^{-\frac{3}{2}}$. Consequently, for n large enough, we get

$$\begin{aligned}
\frac{t_n}{\log t_n}\mathbb{E}_x\left|\int_0^{t_n}\frac{ds}{X_s}-\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right| &\leq C\frac{t_n\sqrt{\Delta_n}}{\log t_n}\left(1+\frac{1}{\sqrt{t_n}}\right) \\
&\leq C\frac{t_n\sqrt{\Delta_n}}{\log t_n}\longrightarrow 0,
\end{aligned}$$

since $\dfrac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)}\rightarrow 0$. Finally, as $\operatorname{diag}(\sqrt{\log t_n},t_n)(\hat{\theta}_{t_n}^{\Delta_n}-\theta)=\operatorname{diag}(\sqrt{\log t_n},t_n)(\hat{\theta}_{t_n}^{\Delta_n}-\hat{\theta}_{t_n})+\operatorname{diag}(\sqrt{\log t_n},t_n)(\hat{\theta}_{t_n}-\theta)$ and thanks to the second assertion of Theorem 6, it is sufficient to prove $\operatorname{diag}(\sqrt{\log t_n},t_n)(\hat{\theta}_{t_n}^{\Delta_n}-\hat{\theta}_{t_n})\longrightarrow 0$ in probability. In order to show this convegence, a similar analysis to that in the previous proof can be done. Indeed, we rewrite the both compenents as follows

$$\begin{aligned}
\log t_n(\hat{a}_{t_n}^{\Delta_n}-\hat{a}_{t_n}) &= \frac{\hat{D}_{t_n}\log t_n(\hat{A}_{t_n}^{\Delta_n}-\hat{A}_{t_n})-\hat{A}_{t_n}\log t_n(\hat{D}_{t_n}^{\Delta_n}-\hat{D}_{t_n})}{\hat{D}_{t_n}^2+\hat{D}_{t_n}(\hat{D}_{t_n}^{\Delta_n}-\hat{D}_{t_n})} \\
t_n(\hat{b}_{t_n}^{\Delta_n}-\hat{b}_{t_n}) &= \frac{\hat{D}_{t_n}t_n(\hat{B}_{t_n}^{\Delta_n}-\hat{B}_{t_n})-\hat{B}_{t_n}t_n(\hat{D}_{t_n}^{\Delta_n}-\hat{D}_{t_n})}{\hat{D}_{t_n}^2+\hat{D}_{t_n}(\hat{D}_{t_n}^{\Delta_n}-\hat{D}_{t_n})},
\end{aligned}$$

where

$$\begin{aligned}
\hat{A}_{t_n} &= \left(\frac{\log X_{t_n}-\log x}{t_n^2}+\frac{\sigma}{t_n^2}\int_0^{t_n}\frac{ds}{X_s}\right)\left(\frac{1}{\log t_n}\int_0^{t_n}X_s ds\right)-\frac{X_{t_n}-x}{t_n\log t_n} \\
\hat{B}_{t_n} &= \frac{\left(\log X_{t_n}-\log x+\sigma\int_0^{t_n}\frac{ds}{X_s}\right)}{t_n\log t_n}-\frac{X_{t_n}-x}{t_n^2}\left(\frac{1}{\log t_n}\int_0^{t_n}\frac{ds}{X_s}\right) \\
\hat{D}_{t_n} &= \left(\frac{1}{\log t_n}\int_0^{t_n}\frac{ds}{X_s}\right)\left(\frac{1}{t_n^2}\int_0^{t_n}X_s ds\right)-\frac{1}{\log t_n},
\end{aligned}$$

and $(\hat{A}_{t_n}^{\Delta_n},\hat{B}_{t_n}^{\Delta_n},\hat{D}_{t_n}^{\Delta_n})$ is simply the discretization version of the triplet $(\hat{A}_{t_n},\hat{B}_{t_n},\hat{D}_{t_n})$. From the above assertions the rate of convergence in probability of $(\hat{A}_{t_n}^{\Delta_n}-\hat{A}_{t_n},\hat{B}_{t_n}^{\Delta_n}-\hat{B}_{t_n},\hat{D}_{t_n}^{\Delta_n}-\hat{D}_{t_n})$ towards zero is t_n, using Theorem 2 we have the convergence in distribution of the couple $(\hat{A}_{t_n},\hat{B}_{t_n},\hat{D}_{t_n})$, and now it is easy to check that $\operatorname{diag}(\sqrt{\log t_n},t_n)(\hat{\theta}_{t_n}^{\Delta_n}-\hat{\theta}_{t_n})$ vanishes. Which completes the proof. □

5 Conclusion

We have investigate the asymptotic behavior of the MLE for square-root diffusions in ergodic and nonergodic cases. However these estimators does not make sense when $a < \sigma$ and are not consistent when $b < 0$. One can wonder: how to overcome this problem by constructing, in these particular cases, new consistent estimators with explicit asymptotic behavior.

References

[1] Mohamed Ben Alaya and Ahmed Kebaier. Parameter Estimation for the Square-root Diffusions : Ergodic and Nonergodic Cases. *hal-00579644 (submitted)*, 2011.

[2] John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[3] Mark Craddock and Kelly A. Lennox. The calculation of expectations for classes of diffusion processes by Lie symmetry methods. *Ann. Appl. Probab.*, 19(1):127–157, 2009.

[4] Éric Fournié and Denis Talay. Application de la statistique des diffusions à un modèle de taux d'intérêt. *Finance*, 12(2):79–111, 1991.

[5] V. Genon-Catalot. Maximum contrast estimation for diffusion processes from discrete observations. *Statistics*, 21(1):99–116, 1990.

[6] Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[7] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic Press Inc., San Diego, CA, sixth edition, 2000. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

[8] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1981.

[9] Jean Jacod. Inference for stochastic process. *Technical Report*, (638), 2001.

[10] Mathieu Kessler. Estimation of an ergodic diffusion from discrete observations. *Scand. J. Statist.*, 24(2):211–229, 1997.

[11] Yury A. Kutoyants. *Statistical inference for ergodic diffusion processes*. Springer Series in Statistics. Springer-Verlag London Ltd., London, 2004.

[12] Damien Lamberton and Bernard Lapeyre. *Introduction au calcul stochastique appliqué à la finance*. Ellipses Édition Marketing, Paris, second edition, 1997.

[13] R. S. Liptser and A. N. Shiryayev. *Statistics of random processes. I,II.* Springer-Verlag, New York, 1977. General theory, Translated by A. B. Aries, Applications of Mathematics, Vol. 5.

[14] Jim Pitman and Marc Yor. Bessel processes and infinitely divisible laws. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 285–370. Springer, Berlin, 1981.

[15] Jim Pitman and Marc Yor. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59(4):425–457, 1982.

[16] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[17] Nakahiro Yoshida. Estimation for diffusion processes from discrete observation. *J. Multivariate Anal.*, 41(2):220–242, 1992.